

Gravitational lensing of charged Ayon-Beato-Garcia black holes and non-linear effects of Maxwell fields

H. Ghaffarnejad¹, M. A. Mojahedi² and H. Niad³

Faculty of Physics, Semnan University, Zip Code 35131-19111, Iran

Abstract

Non-singular Ayon-Beato-Garcia (ABG) spherically symmetric static charged $q = \frac{g}{2m}$ black hole with mass parameter of m is solution of Einstein-nonlinear Maxwell equation. Nonlinear EM fields cause to deviate light geodesics. Hence we must be use an effective ABG metric to study (weak and strong) gravitational lensing of photons. In this work we seek charge effects of the ABG black hole on gravitational lensing of light rays. We use Mathematica software and numerical calculations and set observational data of observed SgrA* black hole. In presence (absence) of nonlinear EM fields the dimensionless photon sphere radius takes maximum value as $x_{ps} = \frac{r_{ps}}{2m} = 7(3)$ for charged $|q| \approx 0.38$ (charge-less Schwarzschild) black hole. In case of weak deflection limits, the deflection angle decreases slower (faster) in presence (absence) of nonlinear counterpart of EM field by raising closest approach distance of bending light rays from the ABG black hole center. Parity of primary (secondary) non-relativistic image is (not) changed by raising source position. Magnification of primary image takes zero minimum value but not for secondary one. In the strong gravitational lensing we obtained that the innermost relativistic image location decreases by increasing $|q|$. Relative distance between innermost and outermost relativistic image location decreases faster (slower) by increasing in absence (presence) of nonlinear EM fields effects. Magnification of outermost relativistic image location takes zero minimum value for $|q| \approx 0.34$ but in absence of nonlinear EM fields effects decreases absolutely by increasing $|q|$.

1 Introduction

Since the advent of Einstein's general relativity theory, black holes and the singularity problem of curved space times become challenging subjects in

¹E-mail address: hghafarnejad@semnan.ac.ir

²E-mail address: amirmojahed@semnan.ac.ir

³E-mail address: niad@semnan.ac.ir

modern physics because of presence of quantum physics. Singularity is the intrinsic character of the most exact solutions of Einstein's equations where Ricci and Kretschmann scalars reach to infinite value at singular point of the space time [1]. Penrose cosmic censorship conjecture states that the causal singularities must be covered by the event horizon and so causes to disconnect interior and exterior regions of the space time [2,3]. However nonsingular metric solutions are also obtained from the Einstein field equation (see for instance [4-23]). In the latter situations the Einstein field equation is coupled to suitable nonlinear electrodynamics fields for which the Ricci and the Kretschmann scalars become regular in whole space time. A good classification of spherically symmetric static regular black holes are collected in ref. [9]. Inspiring a physical central core idea, Bardeen suggested the first spherically symmetric static regular black hole in 1968 containing a horizon without singularity [10]. After his work, other regular black holes were designed based on this model which we call here for instance ABG [11-14], Hayward (HAY) [15] and Neves-Saa (NS) [16,17]. Both solutions of BAR and ABG are asymptotically flat Minkowski and their regularity is controlled via dimensionless charge parameter q . HAY type of regular black hole is obtained by modifying the mass parameter of the BAR black hole. NS type of regular black hole is a HAY type but its asymptotic behavior approaches to a vacuum de Sitter in presence of cosmological constant parameter. Regular black holes are studied also on brane worlds (see [17] and reference therein). The solutions of rotating regular black holes have been introduced in several articles [18-24]. A very important source of strong gravity is the Kerr-Newman-de Sitter (KNDS) and/or Kerr-Newman- anti-de Sitter (KNADS) black hole. Kraniotis studied gravitational lensing of KNDS and KNADS black hole in ref. [25], where closed form analytic solutions of the null geodesics and the gravitational lens equations have been obtained in terms of Appell-Lauricella generalized hypergeometric functions and the elliptic functions of Weierstrass. In these exact solutions all the fundamental parameters of the theory, namely black hole mass, electric charge, rotation and the cosmological constant enter on an equal footing while the electric charge effect on relativistic observable was also investigated. Rotating nonsingular black holes can be treat as natural particle accelerators [24]. Ultra-high energy particle collisions is studied on the regular black holes [26] and backgrounds containing naked singularity [27]. Motion of test particles is studied in regular black hole spacetimes in ref. [28]. Circular geodesics are obtained for BAR and ABG regular black-holes in ref. [29]. The optical effects related to Keplerian discs orbiting

Kehagias-Sfetsos (KS) naked singularities was investigated in ref. [30]. Authors of the latter reference is also mentioned that the close similarity of circular geodesics in KS to the properties of the circular geodesics of the RN naked singularity space times. Schee et al studied also profiled spectral lines generated by keplerian discs orbiting in the Bardeen and ABG spacetimes in ref. [31]. Correspondence between the black holes and the FRW geometries are studied for non-relativistic gravity models in ref. [32]. RN black hole gravitational lensing is studied in ref. [33]. Gravitational lensing from regular black holes is studied in weak deflection limits of light rays [34-36] and in strong deflection limits of light rays [37-41]. Strong deflection limits of light rays can be distinguish gravitational lensing between naked singularity and regular black holes background [41]. There is significant difference between optical phenomena characters of the singular space-times such as SCH, RSN, and non-singular space-times as HAY, BAR, ABG [38]. It is related to the fact that the regular space-times reach to a de Sitter and/or anti-de Sitter like approximately at center $r \rightarrow 0$ (see Eqs. (2.7) and (2.9)). Furthermore we should be point that the nonsingular charged black holes obtained from nonlinear electrodynamics theories in curved space times cause that the photons do not move along null geodesics. As an applicable approach we must be obtain corresponding effective metric for geodesics of moving photons [42-45] and so study their gravitational lensing. The black hole electric charge has also important effects on final state of the Hawking radiation and switching off effects of a quantum evaporating black hole [46]. In this work we study gravitational lensing of light rays moving on the ABG nonsingular black hole in presence of nonlinear EM fields counterparts. The paper is organized as follows.

Briefly, we introduce in section 2 regular ABG black hole metric and its asymptotically behavior against different values of q . In section 3 we calculate effective metric of the ABG black hole for the moving photons by regarding the nonlinear EM fields counterpart. We solve numerically the photon sphere equation of effective metric and obtain photon sphere radius. In section 4 we evaluate general formalism of deflection angle of bending light rays in weak and strong deflection limits. In weak deflection limits we apply the Ohanian lens equation [47] to determine non-relativistic image locations against source positions for observed Sgr A* black hole. Weak deflection angle of bending light rays and their magnifications are evaluated numerically point by point and they are plotted against source location. In the strong deflection limits we use Bozza's formalism [37,38] to obtain logarithmic form

of the deflection angle. We obtain relative distance between innermost and outermost relativistic images and corresponding magnification and then plot their diagrams. Section 5 denotes to concluding remark.

2 ABG spacetime

The ABG spherically symmetric black hole metric defined by Schwarzschild coordinates is [12]

$$ds^2 = -\Omega(r)dt^2 + \frac{dr^2}{\Omega(r)} + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \quad (2.1)$$

with

$$\Omega(r) = 1 - \frac{2mr^2}{(r^2 + g^2)^{3/2}} + \frac{g^2r^2}{(r^2 + g^2)^2} \quad (2.2)$$

and associated electric field

$$F_{tr}(r) = E(r) = gr^4 \left(\frac{r^2 - 5g^2}{(r^2 + g^2)^4} + \frac{15}{2} \frac{m}{(r^2 + g^2)^{\frac{7}{2}}} \right). \quad (2.3)$$

m and g are total mass and electric charge parameters of the black hole respectively. The line element (2.1) is non-singular static solution of Einstein-nonlinear Maxwell equation

$$G_{\mu\nu} = 8\pi T_{\mu\nu} = 8\pi \{ \mathcal{L}_F F_{\mu\eta} F_{\nu}^{\eta} - \mathcal{L} g_{\mu\nu} \}, \quad \mathcal{L}_F = \frac{\partial \mathcal{L}}{\partial F} \quad (2.4)$$

which satisfies the action functional $I = \int dx^4 \left(\frac{R}{16\pi} - \frac{\mathcal{L}(F)}{4\pi} \right)$ where R is Ricci scalar and \mathcal{L} ⁴ is a function of $F = \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$. This metric solution has only the coordinate singularity called as horizon singularity because the Ricci and the Kretschmann scalars become regular at all points of the space time $0 \leq r \leq +\infty$. Defining mass and charge functions as

$$M(r) = m \left(1 + \frac{g^2}{r^2} \right)^{-\frac{3}{2}}, \quad e(r) = g \left(1 + \frac{g^2}{r^2} \right)^{-1} \quad (2.5)$$

⁴Here we do not consider models where \mathcal{L} diverges to infinity in limits $F \rightarrow 0$ for instance lagrangian densities containing logarithmic terms as $\ln[F]$. The latter models can not be studied via perturbation series expansion method.

one can show that the ABG metric (2.1) reduces to a variable mass-charge RN black hole metric as follows.

$$ds^2 = -\left(1 - \frac{2M(r)}{r} + \frac{e^2(r)}{r^2}\right)dt^2 + \frac{dr^2}{\left(1 - \frac{2M(r)}{r} + \frac{e^2(r)}{r^2}\right)} + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \quad (2.6)$$

where $M(\infty) = m$ and $e^2(\infty) = g$ are ADM mass and electric charge viewed from observer located at infinity. Its central region $0 < r < |g|$ behaves as vacuum de Sitter asymptotically:

$$ds^2 \approx -\left(1 - \frac{\Lambda}{3}r^2\right)dt^2 + \frac{dr^2}{\left(1 - \frac{\Lambda}{3}r^2\right)} + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \quad (2.7)$$

for

$$|q| = \frac{g}{2m} < 1 \quad (2.8)$$

and anti de Sitter

$$ds^2 \approx -\left(1 + \frac{\Lambda}{3}r^2\right)dt^2 + \frac{dr^2}{\left(1 + \frac{\Lambda}{3}r^2\right)} + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \quad (2.9)$$

for

$$|q| = \frac{g}{2m} > 1 \quad (2.10)$$

respectively where we defined effective cosmological constant as

$$\Lambda(m, g) = \frac{3(1 - q)}{4m^2q^3}. \quad (2.11)$$

In particular case

$$|q| = \frac{g}{2m} = 1 \quad (2.12)$$

the effective cosmological parameter vanishes $\Lambda = 0$ and so near the center $r \rightarrow 0$, the ABG black hole metric reduces to a flat Minkowski background asymptotically. Furthermore with $g = 0$ the equations (2.5) reads $m = M, e = 0$ which means the ABG regular black hole reduces to a singular charge-less Schwarzschild one. Nonlinear counterpart of the Maxwell stress tensor causes to deviate the photon geodesics where the photons do not

move along the null geodesics. Usually one use an effective metric to study gravitational lensing of the light rays moving on such a charged black holes metric [41-44]. In the following section we seek effective metric of the ABG black hole for photon trajectories.

3 Effective metric for photon trajectories

Assuming $\mathcal{L}(F) = F$, the equation (2.4) leads to the well known linear Einstein-Maxwell gravity where the photon propagates by the null equation

$$g_{\mu\nu}k^\mu k^\nu = 0 \quad (3.1)$$

where k^μ is corresponding four-momentum of the photon, but in general form where $\mathcal{L}(F) \neq F$ the electric field given in the Einstein-nonlinear Maxwell gravity equation (2.4), is self-interacting and so directly is reflected on the photon propagation. In the latter case the photons do not move along null geodesics (3.1) but instead, photons propagate along null geodesics of an effective geometry which depends on used nonlinear theories [43,44,48] as

$$g_{\mu\nu}^{eff}k^\mu k^\nu = 0 \quad (3.2)$$

where

$$g_{\mu\nu}^{eff} = b \left[T_{\mu\nu} - \left(\mathcal{L} + \frac{\mathcal{L}_F^2}{\mathcal{L}_{FF}} + \frac{T^\mu_\mu}{2} \right) g_{\mu\nu} \right] \quad (3.3)$$

in which

$$b = 16 \frac{\mathcal{L}_{FF}}{\mathcal{L}_F} [F^2 \mathcal{L}_{FF}^2 - 16(\mathcal{L}_F + F \mathcal{L}_{FF})^2]^{-1} \quad (3.4)$$

and

$$g_{eff}^{\mu\nu} = \left(\mathcal{L}_F + \frac{\mathcal{L} \mathcal{L}_{FF}}{\mathcal{L}_F} \right) g^{\mu\nu} + \frac{\mathcal{L}_{FF}}{\mathcal{L}_F} T^{\mu\nu}. \quad (3.5)$$

A perturbative approach will be adopted in which nonlinear effects of Maxwell theory have small corrections and up to terms in order of $O(\lambda^2)$ the EM field Lagrangian density $\mathcal{L}(F)$ reads

$$\mathcal{L}(F) \approx F + \frac{\lambda}{2} F^2 \quad (3.6)$$

where self interaction parameter λ satisfies

$$\lambda F \ll 1. \quad (3.7)$$

Applying (3.6) we obtain

$$\mathcal{L}_F \approx 1 + \lambda F, \quad \mathcal{L}_{FF} \approx \lambda, \quad a \approx 1 + 3\lambda F, \quad b \approx -\lambda, \quad T_\mu^\mu \approx 2\lambda F^2 \quad (3.8)$$

$$g_{\mu\nu}^{eff} \approx (1 + 4\lambda F)g_{\mu\nu} - \lambda F_{\mu\eta}F_\nu^\eta \quad (3.9)$$

and

$$g_{eff}^{\mu\nu} \approx (1 + \lambda F)g^{\mu\nu} - \lambda F^{\mu\eta}F_\eta^\nu \quad (3.10)$$

where

$$F = -\frac{E^2(r)}{2} \quad (3.11)$$

and the electric field $E(r)$ is given by (2.3). Inserting (2.1) and (2.3) we obtain exact form of the ABG black hole effective metric (3.10) as follows.

$$ds_{eff}^2 = -f(r)dt^2 + g(r)dr^2 + h(r)(d\theta^2 + \sin^2\theta d\varphi^2) \quad (3.12)$$

where we defined

$$f(r) = (1 - 3\lambda E^2(r))\Omega(r), \quad g(r) = \frac{(1 - 3\lambda E^2(r))}{\Omega(r)}, \quad (3.13)$$

$$h(r) = r^2(1 - 2\lambda E^2(r)).$$

The radius of the event horizon r_H is given by the greatest positive root of the equation $f(r) = 0$. According to study of black hole gravitational lensing, photon sphere construction is one of important characters which must be considered here. It comes from energy condition [49] and is a particular hyper-surface ($r = \text{constant}$) which does not evolve with time. In other words any null geodesic initially tangent to the photon sphere hyper-surface will remain tangent to it. It is made from circulating photons turn turning around the black hole center. Radius of the photon sphere r_{ps} is the greatest positive solution of the equation [48]

$$f(r)\frac{dh(r)}{dr} = h(r)\frac{df(r)}{dr} \quad (3.14)$$

which can be written as

$$\left(\frac{f(r)}{h(r)}\right)'_{|r=r_{ps}} = 0. \quad (3.15)$$

Inserting (3.13) and regarding the condition (3.7) we obtain exact form of the photon sphere equation (3.15) which up to terms in order of $O(\lambda^2)$ reads

$$\left[\left(\frac{\Omega(x)}{x^2} \right)' - 2\epsilon(2m)^2 E(x) E'(x) \left(\frac{\Omega(x)}{x} \right) \right]_{|x=x_{ps}} \simeq 0 \quad (3.16)$$

where $'$ denotes to differentiation with respect to x and we defined

$$x = \frac{r}{2m}, \quad \epsilon = \frac{\lambda}{(2m)^2}, \quad q = \frac{g}{2m}. \quad (3.17)$$

Inserting (2.2) and (2.3) and assuming

$$x_{ps} \simeq x_{ps}^{(0)} + \epsilon x_{ps}^{(1)} \quad (3.18)$$

and up to terms in order of $O(\epsilon^2)$ the photon sphere equation (3.16) leads to the following equations

$$\Delta^2 - 2G_1(x_{ps}^{(0)}, q)\Delta + G_2(x_{ps}^{(0)}, q) = 0 \quad (3.19)$$

and

$$\frac{3(x_{ps}^{(0)})^4}{((x_{ps}^{(0)})^2 + q^2)^{\frac{5}{2}}} - \frac{2(x_{ps}^{(0)})^4 q^2}{((x_{ps}^{(0)})^2 + q^2)^3} = 1. \quad (3.20)$$

where we defined

$$\Delta = \frac{x_{ps}^{(1)}}{x_{ps}^{(0)}}, \quad (3.21)$$

$$G_1(x, q) = 1 - \frac{3}{4(1+q^2)} + \frac{q^2}{2(x^2 + q^2)^{\frac{1}{2}}} - \frac{q^2 x^2}{(x^2 + q^2)^{\frac{3}{2}}} \quad (3.22)$$

and

$$G_2(x, q) = \frac{q^2}{\sqrt{q^2 + x^2}} \left[1 + \frac{5x^6(4q^2 - x^2)}{(x^2 + q^2)^2} + \frac{2x^8(2x^2 - 10q^2 + 15)}{(x^2 + q^2)^3} \right] - \frac{x^6 q^2}{(x^2 + q^2)^2} \left[15 - \frac{5x^2(2x^6 - 11q^2 + 7)}{3(x^2 + q^2)^2} + \frac{x^8(64x^2 - 425q^2)}{8(x^2 + q^2)^3} \right]. \quad (3.23)$$

The equation (3.19) has two roots as $\Delta_{\pm} = G_1 \pm \sqrt{G_1^2 - G_2}$ which its major root makes corrected photon sphere radius (3.18) as follows.

$$x_{ps}^+ \approx x_{ps}^{(0)} \{1 + \epsilon[G_1 + \sqrt{G_1^2 - G_2}]\}. \quad (3.24)$$

We use numerical values given in table 1 and plot diagram of the above equation in figure 1 (see diagram of $x_{ps}^+(\epsilon = 0.1)$). The equation (3.20) has real roots against $(|q|, x_{ps}^{(0)})$ only for $|q| < 0.62$ which we solved numerically via mathematica software and collect major values of $x_{ps}^{(0)}$ into the table 1. Its diagram is given in figure 1 (see diagram of $x_{ps}(\epsilon = 0)$). Inserting Δ_- the equation (3.18) takes some negative values which are forbidden physically (see diagram of $x_{ps}^-(\epsilon = 0.1)$ in figure 1). Hence we restrict our study throughout the paper for $|q| < 1$ where the photon sphere dose not disappear and the central region of the ABG black hole behaves as de Sitter space time (see Eq. (2.7)). In the following section we calculate deflection angle of bending light rays moving on the effective metric (3.12).

4 Deflection angle

When light ray moves at neighborhood of the ABG black hole and deflects without turning around the black hole center then gravitational lensing takes ‘weak deflection limits’ approach. In the latter case closest approach distance of the bending light rays from the black hole center r_0 become larger than the photon sphere radius and two non-relativistic images are usually formed. They are called as primary and secondary images. In general, bending angle of light rays is obtained by solving null geodesics equation defined by (3.12) as follows [50].

$$\alpha(r_0) = I(r_0) - \pi \quad (4.1)$$

where

$$I(r_0) = 2 \int_{r_0 > r_{ps}}^{\infty} \frac{\sqrt{f(r)g(r)/h^2(r)}}{\sqrt{\frac{f(r_0)}{h(r_0)} - \frac{f(r)}{h(r)}}} dr. \quad (4.2)$$

In weak deflection limits we use Taylor series expansion of the integral equation (4.2) to evaluate (4.1). Regarding (3.7) and inserting (3.13) and (3.17), we approximate the integral equation (4.2) which up to terms in order of $O(\epsilon^2)$ become

$$I(x_0) \approx I^{(0)}(x_0) + \epsilon I^{(1)}(x_0) \quad (4.3)$$

where we defined

$$I^{(0)}(x_0) = 2 \int_{x_0}^{\infty} \frac{dx}{x^2 \sqrt{\frac{\Omega(x_0)}{x_0^2} - \frac{\Omega(x)}{x^2}}} \quad (4.4)$$

and

$$I^{(1)}(x_0) = 2(2m)^2 \int_{x_0}^{\infty} \left(\frac{\Omega(x_0)}{x_0^2 x^2} \right) \frac{[E^2(x_0) - E^2(x)]}{\left(\frac{\Omega(x_0)}{x_0^2} - \frac{\Omega(x)}{x^2} \right)^{\frac{3}{2}}} dx. \quad (4.5)$$

Inserting

$$z = \frac{x_0}{x} \quad (4.6)$$

the integral equations (4.4) and (4.5) become respectively

$$I^{(0)}(x_0) = 2 \int_0^1 \frac{dz}{\sqrt{\Omega(x_0) - \Omega(\frac{x_0}{z})z^2}} \quad (4.7)$$

and

$$I^{(1)}(x_0) = 2(2m)^2 \Omega(x_0) \int_0^1 \frac{[E^2(x_0) - E^2(\frac{x_0}{z})]dz}{\left(\Omega(x_0) - \Omega(\frac{x_0}{z})z^2 \right)^{\frac{3}{2}}} \quad (4.8)$$

According to method given in ref. [48], we now expand $E(\frac{x_0}{z})$ and $\Omega(x_0) - \Omega(\frac{x_0}{z})z^2$ in powers of $(1 - z)$ as follows.

$$E\left(\frac{x_0}{z}\right) = E_0 + E_1(1 - z) + E_2(1 - z)^2 + O(3) \quad (4.9)$$

and

$$\Omega(x_0) - \Omega\left(\frac{x_0}{z}\right)z^2 = \Omega_1(1 - z) + \Omega_2(1 - z)^2 + O(3) \quad (4.10)$$

where we defined

$$E_0 = E(x_0), \quad E_1 = x_0 E'(x_0), \quad E_2 = x_0 E'(x_0) + x_0^2 E''(x_0)/2 \quad (4.11)$$

and

$$\Omega_1 = 2\Omega(x_0) - x_0 \Omega'(x_0), \quad \Omega_2 = x_0 \Omega'(x_0) - \Omega(x_0) - x_0^2 \Omega''(x_0)/2. \quad (4.12)$$

Inserting (4.9) and (4.10) the integral equations (4.7) and (4.8) become respectively

$$I^{(0)}(x_0) = 2 \int_0^1 \frac{dz}{\sqrt{\Omega_1(1 - z) + \Omega_2(1 - z)^2}} \quad (4.13)$$

and

$$I^{(1)}(x_0) = \frac{4(2m)^2 \Omega(x_0)}{\Omega_1^{\frac{3}{2}}(x_0)} \int_0^1 \frac{[2E_0 E_1 + (E_1^2 + 2E_0 E_2)u^2 + 2E_1 E_2 u^4 + E_2^2 u^6]du}{(1 - ku^2)^{\frac{3}{2}}} \quad (4.14)$$

where we defined

$$u = \sqrt{1-z}, \quad k = -\frac{\Omega_2}{\Omega_1}. \quad (4.15)$$

The integral equation (4.13) has analytic solution as

$$I^{(0)}(x_0) = -\frac{1}{\sqrt{\Omega_2}} \ln \left[1 + 2\sqrt{\frac{\Omega_2}{\Omega_1}} \left(\sqrt{\frac{\Omega_2}{\Omega_1}} - \sqrt{1 + \frac{\Omega_2}{\Omega_1}} \right) \right] \quad (4.16)$$

but (4.14) has not. We know $\lim_{x_0 \rightarrow \infty} k \approx \frac{1}{2}$ (see Eqs. (4.26) and (4.27)). Thus Taylor series expansion of denominator of the integral equation (4.14) leads to a convergent series for $0 < u < 1$ such that

$$\frac{1}{(1 - ku^2)^{\frac{3}{2}}} \approx 1 + \frac{3}{2}ku^2 + \frac{15}{8}k^2u^4 + \frac{35}{16}k^3u^6 + \frac{315}{128}k^4u^8 + O(u^{10}). \quad (4.17)$$

Inserting (4.17) and $k = -\frac{\Omega_2}{\Omega_1}$ one can obtain series solution of the integral equation (4.14) which up to terms in order of $O(k^5)$ become

$$\begin{aligned} I^{(1)}(x_0) \approx & \frac{2(2m)^2\Omega(x_0)}{\Omega_1^{\frac{3}{2}}(x_0)} \left\{ A(x_0) - B(x_0)\left(\frac{\Omega_2}{\Omega_1}\right) + C(x_0)\left(\frac{\Omega_2}{\Omega_1}\right)^2 \right. \\ & \left. - D(x_0)\left(\frac{\Omega_2}{\Omega_1}\right)^3 + H(x_0)\left(\frac{\Omega_2}{\Omega_1}\right)^4 \right\} \end{aligned} \quad (4.18)$$

where we defined

$$A(x_0) = 2E_0E_1 + \frac{2}{3}E_0E_2 + \frac{1}{3}E_1^2 + \frac{2}{5}E_1E_2 + \frac{1}{7}E_2^2 \quad (4.19)$$

$$B(x_0) = E_0E_1 + \frac{3}{5}E_0E_2 + \frac{3}{10}E_1^2 + \frac{3}{7}E_1E_2 + \frac{1}{6}E_2^2 \quad (4.20)$$

$$C(x_0) = \frac{3}{4}E_0E_1 + \frac{15}{28}E_0E_2 + \frac{15}{56}E_1^2 + \frac{5}{12}E_1E_2 + \frac{15}{88}E_2^2 \quad (4.21)$$

$$D(x_0) = \frac{5}{8}E_0E_1 + \frac{35}{72}E_0E_2 + \frac{35}{144}E_1^2 + \frac{35}{88}E_1E_2 + \frac{35}{208}E_2^2 \quad (4.22)$$

and

$$H(x_0) = \frac{35}{64}E_0E_1 + \frac{315}{704}E_0E_2 + \frac{315}{1408}E_1^2 + \frac{315}{832}E_1E_2 + \frac{21}{128}E_2^2. \quad (4.23)$$

We see that the integral solutions (4.16) and (4.18) diverge to infinite value in limits $\Omega_1 \rightarrow 0$. One can obtain that the equation $\Omega_1 = 0$ describes location of

the photon sphere hypersurface in case of $\epsilon = 0$. To do so we insert $\epsilon = 0$ into the equation (3.16) leading to the relation $\left(\frac{\Omega(x)}{x^2}\right)'_{|x=x_{ps}} = 0$ which is equivalent with

$$\Omega_1(x_{ps}^{(0)}) = 2\Omega(x_{ps}^{(0)}) - x_{ps}^{(0)}\Omega'(x_{ps}^{(0)}) = 0. \quad (4.24)$$

Divergency of $I(x_{ps})$ corresponds to relativistic images angular locations which we will seek in subsection (4.2). In case of $x_0 > x_{ps}$ the term of $\Omega_1(x_0)$ become non-vanishing and so the integral solutions $I^{(0),(1)}(x_0)$ lead to a finite value reaching to non-relativistic images locations. The latter case is called as weak deflection limits of gravitational lensing but the former case is called as strong gravitational lensing. In the following we analyze the integral solutions (4.16) and (4.18) to obtain deflection angle of bending light rays in weak $x_0 \gg x_{ps}$ and strong $x_0 \rightarrow x_{ps}$ deflection limits by using numerical calculation method.

4.1 Weak deflection limits with $x_0 \gg x_{ps}$

It is suitable to obtain asymptotic series expansion of the functions $\Omega(x_0)$, $\Omega_1(x_0)$, $\Omega_2(x_0)$, $E(x_0)$, $E_1(x_0)$, and $E_2(x_0)$, for large values of $x_0 \gg x_{ps}$ which up to terms in order of $O(x_0^4)$ become respectively as

$$\Omega(x_0) \approx 1 - \frac{2}{x_0} + \frac{q^2}{x_0^2} + \frac{3q^2}{x_0^3} \quad (4.25)$$

$$\Omega_1(x_0) \approx 2 - \frac{6}{x_0} + \frac{4q^2}{x_0^2} + \frac{15q^2}{x_0^3} \quad (4.26)$$

$$\Omega_2(x_0) \approx -1 + \frac{2}{x_0} - \frac{6q^2}{x_0^2} - \frac{45q^2}{x_0^3} \quad (4.27)$$

$$(2m)E(x_0) \approx \frac{q}{x_0^2} + \frac{15q}{8x_0^3} \quad (4.28)$$

$$(2m)E_1(x_0) \approx -\frac{2q}{x_0^2} - \frac{45q}{4x_0^3} \quad (4.29)$$

and

$$(2m)E_2(x_0) \approx \frac{q}{x_0^2} - \frac{375q}{4x_0^3}. \quad (4.30)$$

Inserting (4.25), (4.26), (4.27), (4.28), (4.29) and (4.30) one can obtain asymptotic behavior of the functions $A(x_0)$, $B(x_0)$, $C(x_0)$, $D(x_0)$, $H(x_0)$ and $k(x_0)$ respectively as follows.

$$A(x_0) \approx \frac{93}{140} \frac{q^2}{m^2 x_0^4} - \frac{911}{112} \frac{q^2}{m^2 x_0^5} + \frac{87375}{224} \frac{q^2}{m^2 x_0^6} \quad (4.31)$$

$$B(x_0) \approx -\frac{187}{840} \frac{q^2}{m^2 x_0^4} - \frac{691}{224} \frac{q^2}{m^2 x_0^5} + \frac{204765}{448} \frac{q^2}{m^2 x_0^6} \quad (4.32)$$

$$C(x_0) \approx -\frac{1027}{7392} \frac{q^2}{m^2 x_0^4} - \frac{17095}{9856} \frac{q^2}{m^2 x_0^5} + \frac{9173475}{19712} \frac{q^2}{m^2 x_0^6} \quad (4.33)$$

$$D(x_0) \approx -\frac{2875}{27456} \frac{q^2}{m^2 x_0^4} - \frac{124925}{109824} \frac{q^2}{m^2 x_0^5} + \frac{33506125}{73216} \frac{q^2}{m^2 x_0^6} \quad (4.34)$$

$$H(x_0) \approx -\frac{6307}{73216} \frac{q^2}{m^2 x_0^4} - \frac{239925}{292864} \frac{q^2}{m^2 x_0^5} + \frac{260560125}{585728} \frac{q^2}{m^2 x_0^6} \quad (4.35)$$

$$k(x_0) \approx \frac{1}{2} + \frac{1}{2x_0} + \frac{(4q^2 + 3)}{2x_0^2} + \frac{(95q^2 + 18)}{4x_0^3}. \quad (4.36)$$

Inserting (4.25)-(4.36) one can obtain asymptotic series expansion form of the integral equations (4.16) and (4.18) for $x_0 \gg x_{ps}$ respectively as follows.

$$\begin{aligned} I^{(0)}(x_0) &\simeq \sqrt{2} + \frac{1.80\sqrt{2}}{x_0} + (4.40 - 0.57q^2) \frac{\sqrt{2}}{x_0^2} + (11.88 - 1.77q^2) \frac{\sqrt{2}}{x_0^3} \\ &+ (30.21 - 23.05q^2 + 1.65q^4) \frac{\sqrt{2}}{x_0^4} + (27.59q^4 - 99.00q^2 + 81.15) \frac{\sqrt{2}}{x_0^5} \\ &+ (39.64 + 91.27q^2 + 28.31q^4 + 0.59q^6) \frac{\sqrt{2}}{x_0^6} \end{aligned} \quad (4.37)$$

and

$$I^{(1)}(x_0) \simeq -\frac{1.66\sqrt{2}q^2}{x_0^4} - \frac{24.23\sqrt{2}q^2}{x_0^5} + \frac{(1.38q^2 + 1566.4)\sqrt{2}q^2}{x_0^6} \quad (4.38)$$

where deflection angle (4.1) become

$$\alpha_{weak}(x_0) = -\pi + I^{(0)}(x_0) + \epsilon I^{(1)}(x_0). \quad (4.39)$$

Setting $\epsilon = 0; 0.1$ and sample charge $|q| = 0.34$ we plot diagram of the above weak deflection angle against x_0 in figure 1. This diagram shows decrease of α_{weak} by raising x_0 as slower (faster) in presence (absence) of nonlinear EM fields counterpart. Also direction of moving light rays is changed from right side to left side of the ABG lens at $x_0 \approx 3.805(3.810)$ in absence (presence) of nonlinear EM field counterpart. Furthermore we obtain $\lim_{x_0 \rightarrow \infty} \alpha_{weak} \approx -1.73 \text{Radian}$. In the next subsection we proceed to determine non-relativistic image locations.

4.1.1 Non-relativistic images locations

In order to calculate the weak deflection images we choose Ohanian lens equation [47] which has high accuracy and so lower errors with respect to other lens equations [51]. It has the advantage of being the closest relative of the exact lens equation, since it only contains the asymptotic approximation without any additional assumptions. It can be rewritten against observational coordinates as image position θ , source position β and deflection angle of bending light rays α_{weak} as follows (see [51] for more discussions).

$$\arcsin(D_L \sin \theta) - \arcsin(D_S \sin \beta) = \alpha_{weak} - \theta \quad (4.40)$$

in which we defined

$$D_L = \frac{d_{OL}}{d_{LS}}, \quad D_S = \frac{d_{OS}}{d_{LS}}. \quad (4.41)$$

In the above equations d_{OS} is distance between observer and source, d_{OL} is distance between the observer and the lens, d_{LS} is distance between the lens and the source. θ is formed when a line passing through the observer and the image is coincide optical axis (line passing through the observer and the lens). β is formed when a line passing through the observer and the source is coincide the optical axis. Applying generic gravitational lensing configuration given in figure 1 of ref. [51], we obtain identities

$$d_{OS} \cos \beta = d_{OL} + d_{LS} \cos \gamma \quad (4.42)$$

and

$$d_{OS} \sin \beta = d_{LS} \sin \gamma \quad (4.43)$$

where the angle of $\pi - \gamma$ is made by crossing d_{OL} and d_{LS} . Eliminating γ term between (4.42) and (4.43) and using (4.41) one can obtain

$$D_S^\pm = D_L \cos \beta \pm \sqrt{1 - D_L^2 \sin^2 \beta}. \quad (4.44)$$

Inserting (4.44), the lens equation (4.40) reads

$$\begin{aligned} \arcsin(D_L \sin \theta) - \arcsin\left(\frac{D_L \sin 2\beta}{2} \pm \sqrt{\sin^2 \beta - D_L^2 \sin^4 \beta}\right) - \theta \\ = \alpha_{weak} \end{aligned} \quad (4.45)$$

where D_L must be inserted via experimental dates. For weak gravitational lensing we can use Taylor series expansion of the lens equation (4.45) for neighborhoods $(\beta, \theta) \rightarrow (0, 0)$ which up to terms in order of $O(7)$ become

$$\theta_p^* - \beta^* + 0.17(\theta_p^{*3} - \beta^{*3}) + 0.075(\theta_p^{*5} - \beta^{*5}) \simeq -1.73 \quad (4.46)$$

where the subscript p denotes to the word ‘primary’ and we insert $\lim_{x_0 \rightarrow \infty} \alpha_{weak}(x_0) = \sqrt{2} - \pi = -1.73$ and definitions

$$\beta^* = D_L \beta, \quad \theta^* = D_L \theta. \quad (4.47)$$

As a realistic example of gravitational lens we consider a big black hole located in the center of Galaxy and study image locations of a star located far from it. This black hole is called as Sgr A* [52,53]. Its mass is estimated $3.6 \times 10^6 M_\odot$ and its distance from the earth is $d_{OL} = 8 \text{ kpc} = 2.47 \times 10^{17} m$ with corresponding Schwarzschild radius $R_{SCH} = 10^{10} m$. We consider a source to be a star located at distance $d_{LS} = 1.7 \times 10^{13} m$ from the black hole which is far from the margin of the accretion disk of the black hole, so it may not be fall toward the black hole center. For the latter black hole we can use approximation

$$D_L \approx 1.45 \times 10^4 \quad (4.48)$$

for which $\beta = \frac{\beta^*}{1.45 \times 10^4}$ become more small angle in ‘Radian’ units but in units of arc-second become $\beta_{arcsec} \approx 0.25 \beta_{arcsec}^*$. The lens equation (4.46) determines primary non-relativistic image locations by fixing β^* and α_{weak} and D_L . Secondary non-relativistic image location $\theta^s(\beta) = \theta^p(-\beta)$ is obtained by inserting $\beta \rightarrow -\beta$ into the lens equation (4.46) as follows.

$$\theta_s^* + \beta^* + 0.17(\theta_s^{*3} + \beta^{*3}) + 0.075(\theta_s^{*5} + \beta^{*5}) \simeq -1.73 \quad (4.49)$$

where the subscript s denotes to the ‘secondary’ word. We calculate numerical values of θ_{ps}^* for ansatz $0 < \beta < \frac{\pi}{2}$ and collect them into the table 2. We plot diagrams of the lens equations (4.46) and (4.49) in figure 1. This shows that the primary (secondary) non-relativistic image location raises by increasing β^* with (without) change of its parity. We now proceed to study corresponding magnifications as follows.

4.1.2 Magnifications

The magnification μ of an image is defined as the ratio of flux of the image to flux of un-lensed source. It has two components called as tangential $\mu_t = \frac{\sin \theta}{\sin \beta}$ and radial $\mu_r = \frac{d\theta}{d\beta}$ which their multiplication makes the magnification as

$$\mu = \left| \frac{\sin \beta}{\sin \theta} \frac{d\beta}{d\theta} \right|^{-1}. \quad (4.50)$$

In case of weak deflection limits we can use $\sin \theta \approx \theta$ and $\sin \beta \approx \beta$ to approximate as follows.

$$\mu_{weak} \approx \left| \frac{\beta}{\theta} \frac{d\beta}{d\theta} \right|^{-1} \quad (4.51)$$

Inserting (4.46) the magnification (4.51) reads

$$\mu_{weak}^p(\beta^*, \theta^*) \approx \left| \frac{\theta^{*-1} + 0.51\theta^* + 0.285\theta^{*3}}{\beta^{*-1} + 0.51\beta^* + 0.285\beta^{*3}} \right|^{-1} \quad (4.52)$$

in which superscript p denotes to primary images $\theta^p(\beta)$ magnifications with positive parity. Magnifications of the secondary images $\theta^s(\beta) = \theta^p(-\beta)$ with negative parity is obtained by inserting (4.49) into the equation (4.51) as $\mu_{weak}^s(\beta) = \mu_{weak}^p(-\beta)$. Total magnification μ_{tot} and its weighted-centroid μ_{cent} are two important quantities which are used to micro-lensing state defined by respectively

$$\mu_{tot} = |\mu_s| + |\mu_p| \quad (4.53)$$

and

$$\mu_{cent} = \frac{\theta_p |\mu_p| + \theta_s |\mu_s|}{|\mu_p| - |\mu_s|}. \quad (4.54)$$

Inserting numerical values given in the table 2 we calculate numerical values of $\mu^{p,s}, \mu_{tot}, \mu_{cent}$ to plot their diagrams against β^* in figure 2. Diagrams show that μ_p (μ_{tot}) has a minimum (non-) zero value for $\beta^* \approx 1.26$ by raising β^* . μ_s decreases absolutely by increasing β^* . μ_{cent} diverge to infinite value for $\beta^* \approx 1.0$. We now proceed to study strong deflection limits of the gravitational lensing for $x_0 \rightarrow x_{ps}$.

4.2 Strong deflection limits with $r_0 \approx r_{ps}$

Defining

$$y = \frac{x_0}{x_{ps}} - 1, \quad 0 < y < 1 \quad (4.55)$$

and using (3.17) and (3.20) and inserting Taylor series expansion of the integral solutions (4.16) and (4.18), the deflection angle (4.1) at neighborhood of the photon sphere become

$$\alpha_{strong}(y) \approx \eta_0(x_{ps}^{(0)}) - \eta_1(x_{ps}^{(0)}) \ln y + \frac{\epsilon}{y^{\frac{3}{2}}} \left[\eta_2(x_{ps}^{(0)}) + \frac{\eta_3(x_{ps}^{(0)})}{y} + \frac{\eta_4(x_{ps}^{(0)})}{y^2} + \frac{\eta_5(x_{ps}^{(0)})}{y^3} + \frac{\eta_6(x_{ps}^{(0)})}{y^4} \right] \quad (4.56)$$

where we defined

$$\eta_0(x) = -\pi + \frac{1}{\sqrt{\Omega_2(x)}} \ln \left(\frac{4\Omega_2(x)}{xF(x)} \right) \quad (4.57)$$

$$\eta_1(x) = \frac{1}{\sqrt{\Omega_2(x)}} \quad (4.58)$$

$$\eta_2(x) = \frac{2(2m)^2 A(x) \Omega(x)}{(xF(x))^{\frac{3}{2}}} \quad (4.59)$$

$$\eta_3(x) = -\frac{2(2m)^2 B(x) \Omega(x) \Omega_2(x)}{(xF(x))^{\frac{5}{2}}} \quad (4.60)$$

$$\eta_4(x) = \frac{2(2m)^2 C(x) \Omega(x) \Omega_2^2(x)}{(xF(x))^{\frac{7}{2}}} \quad (4.61)$$

$$\eta_5(x) = -\frac{2(2m)^2 D(x) \Omega(x) \Omega_2^3(x)}{(xF(x))^{\frac{9}{2}}} \quad (4.62)$$

$$\eta_6(x) = \frac{2(2m)^2 H(x) \Omega(x) \Omega_2^4(x)}{(xF(x))^{\frac{11}{2}}} \quad (4.63)$$

and

$$F(x) = \frac{30x^5}{(x^2 + q^2)^{\frac{7}{2}}} - \frac{24x^3}{(x^2 + q^2)^{\frac{5}{2}}} - \frac{24q^2 x^5}{(x^2 + q^2)^4} + \frac{16q^2 x^3}{(x^2 + q^2)^3}. \quad (4.64)$$

We know that y is coordinate dependent. It is useful to write the deflection angle (4.56) against coordinate independent quantity $u_0 = u(x_0) = \frac{2mx_0}{\sqrt{\Omega(x_0)}}$ which is minimum dimensionless impact parameter of the bending light rays.

To do so we must obtain its Taylor series expansion around $x_0 \rightarrow x_{ps}$ which up to second order term become

$$\rho - 1 \approx G(x_{ps}^{(0)})y, \quad \rho = \frac{u_0}{u_{ps}(x_{ps}^{(0)})} \quad (4.65)$$

where

$$G(x) = \frac{x^6 + 5x^4q^2 + 3x^2q^4 + q^6 - 3x^4\sqrt{x^2 + q^2}}{x^6 + 4x^4q^2 + 4x^2q^4 + q^6 - 2(x^2 + q^2)\sqrt{x^2 + q^2}} \quad (4.66)$$

and

$$\frac{u_{ps}(x)}{2m} = \frac{x(x^2 + q^2)}{\sqrt{x^4 + 3q^2x^2 + q^4 - 2x^2\sqrt{x^2 + q^2}}}. \quad (4.67)$$

Inserting (4.65) the deflection angle (4.56) reads

$$\alpha_s(\rho) \approx \gamma_0 - \gamma_1 \ln(\rho - 1) + \frac{\epsilon}{(\rho - 1)^{\frac{3}{2}}} \left[\gamma_2 + \frac{\gamma_3}{(\rho - 1)} + \frac{\gamma_4}{(\rho - 1)^2} + \frac{\gamma_5}{(\rho - 1)^3} + \frac{\gamma_6}{(\rho - 1)^4} \right] \quad (4.68)$$

where

$$\begin{aligned} \gamma_0 &= \eta_0 + \eta_1 \ln G, \quad \gamma_1 = \eta_1, \quad \gamma_2 = \eta_2 G^{\frac{3}{2}}, \quad \gamma_3 = \eta_3 G^{\frac{5}{2}}, \\ \gamma_4 &= \eta_4 G^{\frac{7}{2}}, \quad \gamma_5 = \eta_5 G^{\frac{9}{2}}, \quad \gamma_6 = \eta_6 G^{\frac{11}{2}}. \end{aligned} \quad (4.69)$$

Inserting numerical values of the quantities $(q, x_{ps}^{(0)})$ from the table 1 for $\epsilon = 0; 0.1$ we can calculate exact form of the function $\alpha_s(\rho)$. However the solution (4.68) diverges to infinite value for $\rho \rightarrow 1$ (the photon sphere hypersurface) corresponding to infinite number of relativistic images where the bending light rays circulate around the lens center before than that achieve to the observer. In the following section we seek situations where some relativistic images are formed.

4.2.1 Relativistic images and magnifications

When the source, the lens and the observer become strictly aligned, β and θ take small values and so the lens equation approaches to the following form (see Eq. 32 in ref. [48]).

$$\beta = \theta - D_{SL} \Delta \alpha_n \quad (4.70)$$

where we defined

$$D_{SL} = \frac{d_{OS}}{d_{OL}} = \frac{D_S}{D_L} \quad (4.71)$$

and

$$\Delta\alpha_n = \alpha_n - 2n\pi. \quad (4.72)$$

β , θ_n and $0 < \Delta\alpha_n < 1$ describe angular locations of the source, the n^{th} relativistic image and the deflection angle respectively with $n = 0, \pm 1, \pm 2, \dots$. Non-relativistic images are determined by setting $n = 0$ and relativistic images with positive (negative) parity are determined by setting $n = 1, 2, \dots$ ($n = -1, -2, \dots$) where one can obtain $\Delta\alpha_{-n} = 2\alpha - \Delta\alpha_n$. In case of retro-lensing where observer is located between source and lens, the light rays come back after than that turning around the lens (see figure 1 at ref. [40,41]). In the latter case the parameter $2n$ given in the formula (4.72) must be replaced with $2n - 1$. Using (4.70) and (4.72) the relativistic images lens equation become

$$\alpha = 2n\pi + \frac{(\theta_n - \beta)}{D_{LS}} \quad (4.73)$$

which by inserting (4.68) reads

$$\begin{aligned} \beta^* &= 2n\pi + \theta_n^* - \gamma_0 + \gamma_1 \ln \left(\frac{\theta_n^*}{\theta_\infty^*} - 1 \right) \\ &- \frac{\epsilon}{\left(\frac{\theta_n^*}{\theta_\infty^*} - 1 \right)^{\frac{3}{2}}} \left[\gamma_2 + \frac{\gamma_3}{\left(\frac{\theta_n^*}{\theta_\infty^*} - 1 \right)} + \frac{\gamma_4}{\left(\frac{\theta_n^*}{\theta_\infty^*} - 1 \right)^2} + \frac{\gamma_5}{\left(\frac{\theta_n^*}{\theta_\infty^*} - 1 \right)^3} + \frac{\gamma_6}{\left(\frac{\theta_n^*}{\theta_\infty^*} - 1 \right)^4} \right] \end{aligned} \quad (4.74)$$

where we defined

$$\beta^* = \frac{\beta}{D_{LS}}, \quad \theta_n^* = \frac{\theta_n}{D_{LS}}, \quad \theta_\infty^* = \frac{\theta_\infty}{D_{LS}} = \frac{2mu_{ps}(x_{ps}^{(0)})}{d_{OS}}, \quad \theta_\infty = \frac{2mu_{ps}(x_{ps}^{(0)})}{d_{OL}} \quad (4.75)$$

When the source, the lens and the observer become strictly aligned, then we can approximate $d_{OL} + d_{LS} \approx d_{OS}$ which for observation date Sgr A* as $d_{OL} = 2.47 \times 10^{17}m$, $d_{LS} = 1.7 \times 10^{13}m$, $R_{Sch} = 2m \approx 10^{10}m$ we will have $d_{OS} \approx d_{OL}$ and so $\theta_\infty^*(Radian) \approx 4 \times 10^{-8}u_{ps}(x_{ps}^{(0)})$ and by multiplying $3600''$ reads

$$\theta_\infty^*(\mu arcsec) \approx 144u_{ps}(x_{ps}^{(0)}). \quad (4.76)$$

The above equation describes innermost relativistic image location (see figure 3). Up to negligible term of $\Delta\alpha_n = \theta_n^* - \beta^*$ and inserting the approximation

$$\theta_n^* \approx \theta_n^{*(0)} + \epsilon\theta_n^{*(1)} \quad (4.77)$$

the lens equation (4.73) leads to the following relations.

$$-2n\pi + \gamma_0 - \gamma_1 \ln \left(\frac{\theta_n^{*(0)}}{\theta_\infty^*} - 1 \right) = 0 \quad (4.78)$$

and

$$\frac{\theta_n^{*(1)}}{\theta_\infty^*} = \frac{\gamma_1^{-1}}{\left(\frac{\theta_n^{*(0)}}{\theta_\infty^*} - 1 \right)^{\frac{1}{2}}} \left[\gamma_2 + \frac{\gamma_3}{\left(\frac{\theta_n^{*(0)}}{\theta_\infty^*} - 1 \right)} + \frac{\gamma_4}{\left(\frac{\theta_n^{*(0)}}{\theta_\infty^*} - 1 \right)^2} + \frac{\gamma_5}{\left(\frac{\theta_n^{*(0)}}{\theta_\infty^*} - 1 \right)^3} + \frac{\gamma_6}{\left(\frac{\theta_n^{*(0)}}{\theta_\infty^*} - 1 \right)^4} \right]. \quad (4.79)$$

Solving (4.78) and (4.79) one can obtain

$$\xi_n = \frac{\theta_n^{*(0)}}{\theta_\infty^*} = 1 + e^{\frac{(\gamma_0 - 2n\pi)}{\gamma_1}} \quad (4.80)$$

and

$$\zeta_n = \frac{\theta_n^{*(1)}}{\theta_\infty^*} = \frac{1}{\gamma_1} \left[\gamma_2 e^{\frac{(\gamma_0 - 2n\pi)}{2\gamma_1}} + \gamma_3 e^{\frac{(\gamma_0 - 2n\pi)}{\gamma_1}} + \gamma_4 e^{\frac{2(\gamma_0 - 2n\pi)}{\gamma_1}} + \gamma_5 e^{\frac{3(\gamma_0 - 2n\pi)}{\gamma_1}} + \gamma_6 e^{\frac{4(\gamma_0 - 2n\pi)}{\gamma_1}} \right] \quad (4.81)$$

Relativistic Einstein's rings θ_n^E are obtained by setting $\beta = 0$ and solving (4.73). We now are in position to calculate n^{th} relativistic images magnification which is determined by

$$\mu_n \approx \left| \frac{\beta^* d\beta^*}{\theta_n^* d\theta_n^*} \right|^{-1}. \quad (4.82)$$

Inserting (4.74) together with some simple calculations we obtain exact form of the magnification (4.82) which up to terms in order of $O(\epsilon^2)$ become

$$\mu_n \approx \left| \lambda_n^{(0)} + \epsilon \lambda_n^{(1)} \right|^{-1} \quad (4.83)$$

where we defined

$$\lambda_n^{(0)} = \left(1 + \frac{\gamma_1}{\xi_n - 1} \right) \left(1 + \frac{2n\pi - \gamma_0}{\theta_\infty^* \xi_n} + \frac{\gamma_1}{\xi_n} \ln(\xi_n - 1) \right) \quad (4.84)$$

and

$$\lambda_n^{(1)} = \left(1 + \frac{2n\pi - \gamma_0}{\theta_\infty^* \xi_n} + \frac{\gamma_1}{u \xi_n} \ln(\xi_n - 1) \right) \left[-\frac{\gamma_1 \zeta_n}{(\xi_n - 1)^2} + \frac{3}{2} \gamma_2 (\xi_n - 1)^{-\frac{5}{2}} + \frac{5}{2} \gamma_3 (\xi_n - 1)^{-\frac{7}{2}} \right]$$

$$\begin{aligned}
& + \frac{7}{2}\gamma_4(\xi_n - 1)^{-\frac{9}{2}} + \frac{9}{2}\gamma_5(\xi_n - 1)^{-\frac{11}{2}} + \frac{11}{2}\gamma_6(\xi_n - 1)^{-\frac{13}{2}} \Big] + \left(1 + \frac{\gamma_1}{\xi_n - 1}\right) \\
& \times \left[\frac{\zeta_n(\gamma_0 - 2n\pi)}{\theta_\infty^* \xi_n^2} - \frac{\gamma_1 \zeta_n}{\xi_n^2} \ln(\xi_n - 1) + \frac{\gamma_1 \zeta_n}{\xi_n(\xi_n - 1)} + \frac{\gamma_2}{\xi_n}(\xi_n - 1)^{-\frac{3}{2}} + \frac{\gamma_3}{\xi_n}(\xi_n - 1)^{-\frac{5}{2}} \right. \\
& \quad \left. + \frac{\gamma_4}{\xi_n}(\xi_n - 1)^{-\frac{7}{2}} + \frac{\gamma_5}{\xi_n}(\xi_n - 1)^{-\frac{9}{2}} + \frac{\gamma_6}{\xi_n}(\xi_n - 1)^{-\frac{11}{2}} \right]. \quad (4.85)
\end{aligned}$$

We now assume a simplest situation where the outermost relativistic image θ_1^* is only resolved as a single image, while all the remaining ones are packed together at θ_∞^* . In the latter case relative angular distance between outermost θ_1^* and innermost θ_∞^* relativistic image location become

$$s = \theta_1^* - \theta_\infty^* = s^{(0)} + \epsilon s^{(1)} \quad (4.86)$$

where we defined

$$s^{(0)} = \theta_\infty^* e^{\frac{\gamma_0 - 2\pi}{\gamma_1}} \quad (4.87)$$

and

$$s^{(1)} = \frac{\theta_\infty^*}{\gamma_1} \left[\gamma_2 e^{\frac{\gamma_0 - 2\pi}{2\gamma_1}} + \gamma_3 e^{\frac{\gamma_0 - 2\pi}{\gamma_1}} + \gamma_4 e^{\frac{2(\gamma_0 - 2\pi)}{\gamma_1}} + \gamma_5 e^{\frac{3(\gamma_0 - 2\pi)}{\gamma_1}} + \gamma_6 e^{\frac{4(\gamma_0 - 2\pi)}{\gamma_1}} \right] \quad (4.88)$$

for which one can write relative magnification r of the outermost relativistic image flux with respect to flux of the all packed ones at θ_∞^* as

$$r = \frac{\mu_1}{\sum_{n=2}^{\infty} \mu_n}. \quad (4.89)$$

Inserting (4.83) the above equation reads

$$r = \frac{\left| \lambda_1^{(0)} + \epsilon \lambda_1^{(1)} \right|^{-1}}{\sum_{n=2}^{\infty} \left| \lambda_n^{(0)} + \epsilon \lambda_n^{(1)} \right|^{-1}}. \quad (4.90)$$

Just we insert numerical values of $(|q|, x_{ps}^{(0)})$ from the table 1 then we will be in position to determine the strong gravitational lensing parameters $\gamma_i; i = 0, 1, 2, \dots, 6$, s and r as follows. We plot diagrams of θ_∞^* , s and μ_1 in figure 3 against $|q|$ in absence $\epsilon = 0$ and presence $\epsilon = 0.1$ of the nonlinear counterpart

of the EM fields for observed large black hole Sgr A*. Mathematica software needs more times to calculate denominator of the equation (4.90) and so we restrict us to plot μ_1 instead to r . In the Bozza's formalism $\gamma_{0,1}$ are important parameters to study their behavior in the strong gravitational lensing [48]. Hence we plot $\gamma_i; i = 0, \dots, 6$ against $|q|$ in figure 4. These diagrams show major effects of $\gamma_{0,1}$ vs $\gamma_{2,3,4,5,6}$. Also one can look the figure 4 where $\gamma_{0,2,4,6}$ decrease by raising $|q|$ but $\gamma_{1,3,5}$ increase.

5 Concluding remark

We studied black hole charge effects and nonlinear effects of interacting Maxwell fields on weak and strong gravitational lensing from ABG black hole. Using experimental characteristics of observed Sgr A* black hole, we apply Mathematica software to calculate numerical values of deflection angle of bending light rays, image positions and magnifications in weak and strong deflection limits. Our calculations restrict to choose $|q| < 1$ where the photon sphere is not destruct and central region of the ABG black hole behaves as de Sitter space time. Our work predicts raise of the photon sphere radius by increasing the electric charge in presence of nonlinear counterpart of EM fields as $x_{ps} > 3$. While in absence of the nonlinear EM field effects maximum value of the photon sphere radius belongs to the charge-less Schwarzschild black hole as $x_{ps} = 3$. Weak deflection angle decreases slower (faster) by increasing of closest distance of bending light rays from the black hole center in presence (absence) of nonlinear counter part of EM fields. Parity of primary (secondary) non-relativistic image is (not) changed by raising the source position. Magnification of the primary image exhibit with zero minimum value but not for secondary one. Total magnification takes a non-vanishing minimum value at particular source position and centroid one exhibit with divergency. In the strong deflection limit of gravitational lensing we obtained that the innermost relativistic image location decreases by increasing the electric charge value. Relative angular distance between innermost and outermost relativistic image decreases by increasing the charge effect. Nonlinear effects of the EM fields have more small corrections. Magnification of the outermost relativistic image increase by raising the charge in absence of the nonlinear effects of EM fields but in presence of the latter effects it take a zero minimum value for a particular charge value. As extension of our work we will study same work in case of $|q| > 1$ where the photon sphere hypersurface

disappears and so central region of the ABG black hole behaves as anti-de Sitter space time in the future work. Also as an other applicable work one can seek same proposal by using singular lagrangian of the nonlinear EM fields effects.

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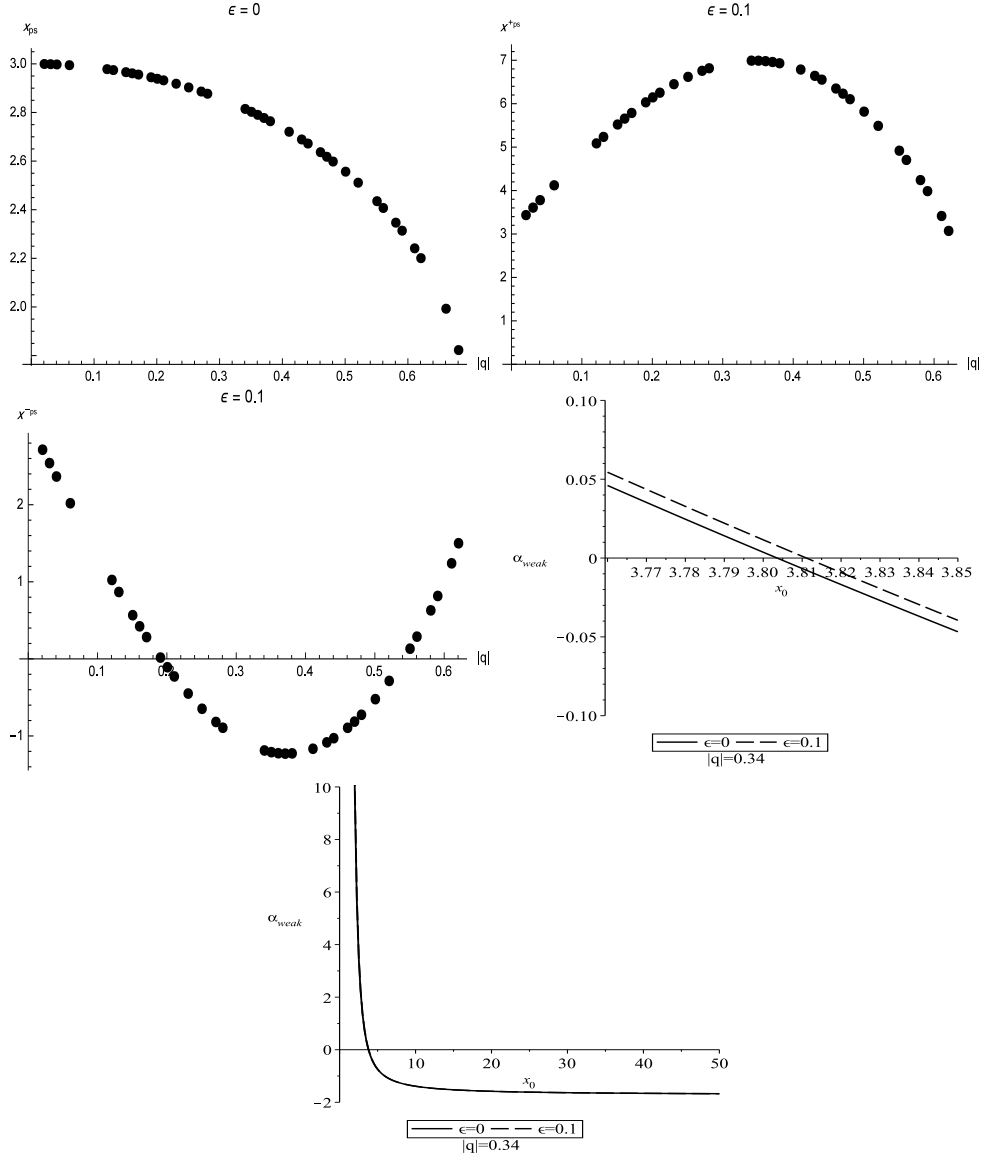


Figure 1: Diagrams of $x_{ps}^{(0)}, x_{ps}^{\pm}$ are plotted against $|q|$ and α_{weak} against x_0 .

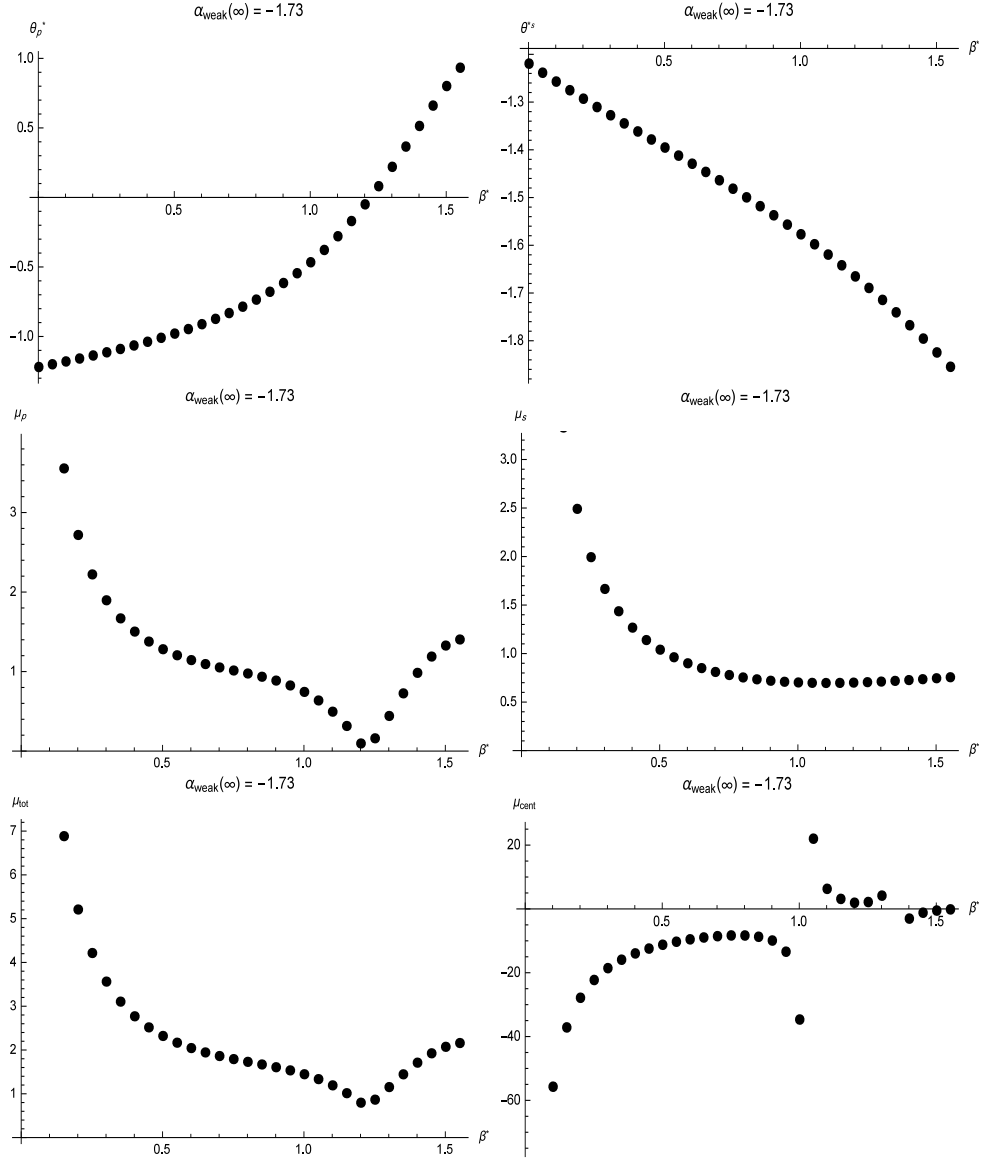


Figure 2: Diagrams of $\theta_{p,s}^*, \mu_{p,s}, \mu_{\text{tot}}, \mu_{\text{cent}}$ are plotted against β^*

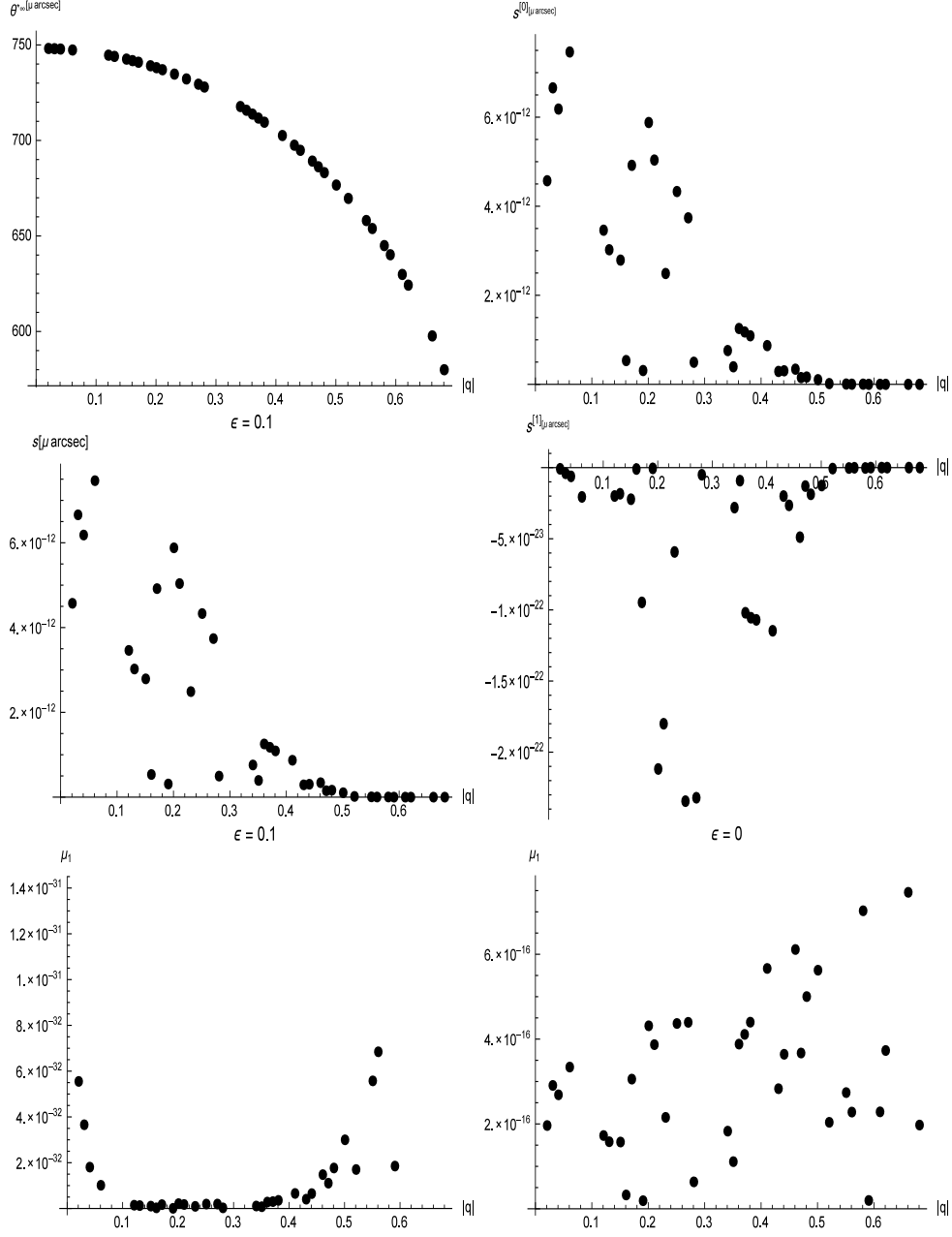


Figure 3: Diagrams of θ_{∞}^* , μ_1 and s are plotted against $|q|$ for $\epsilon = 0; 0.1$

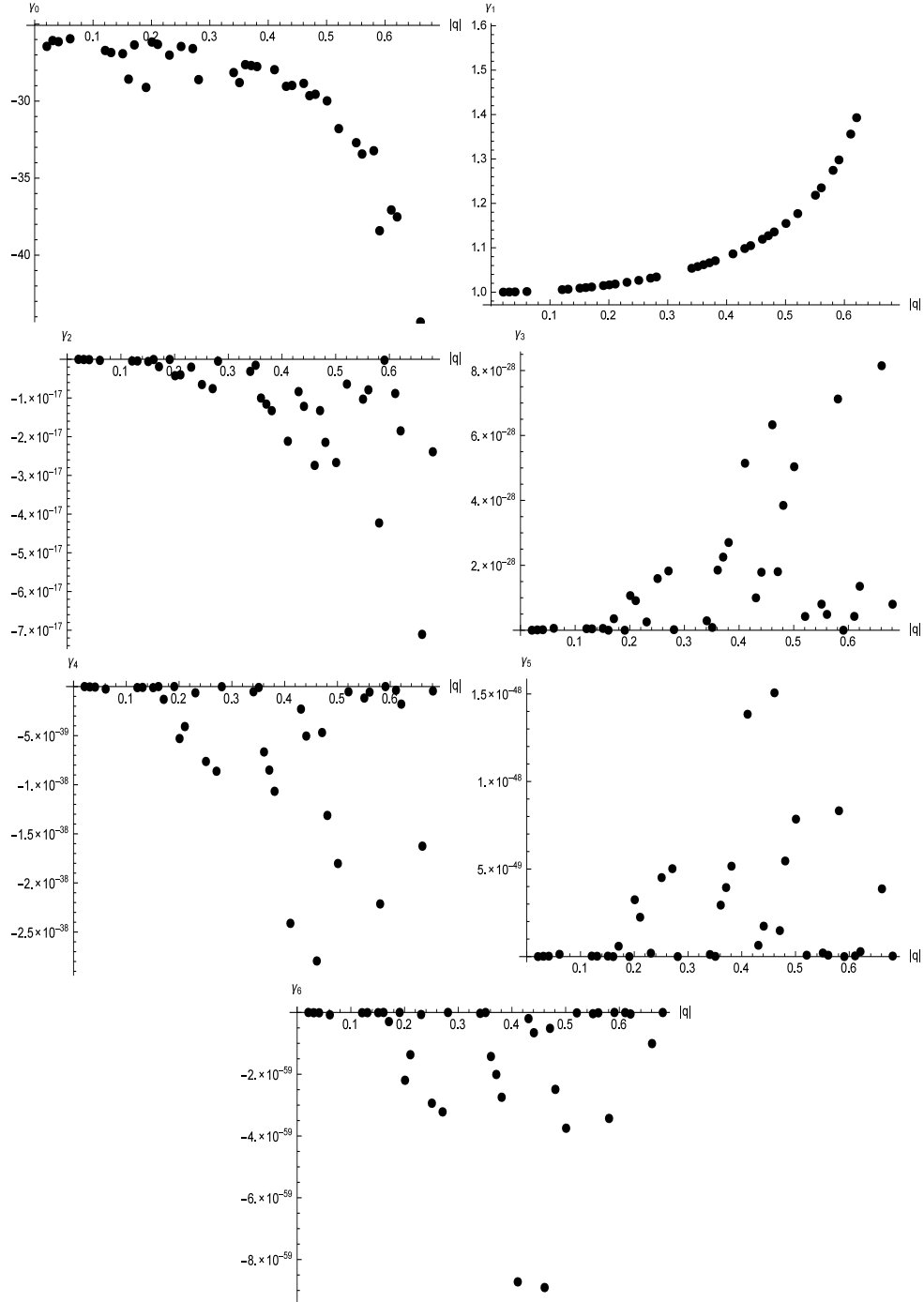


Figure 4: Diagrams of $\gamma_i; i = 0, 1, 2, \dots, 6$ are plotted against $|q|$.

Table 1. Numerical real roots of the photon sphere equation (3.18).

$ q , x_{ps}^{(0)}$	$ q , x_{ps}^{(0)}$	$ q , x_{ps}^{(0)}$
0.01 , 2.999849992	0.21 , 2.932253768	0.44 , 2.672202403
0.02 , 2.999399875	0.23 , 2.918330121	0.46 , 2.636644194
0.03 , 2.998649366	0.25 , 2.902975688	0.47 , 2.617801576
0.04 , 2.997597994	0.27 , 2.886140632	0.48 , 2.598198581
0.06 , 2.994589822	0.28 , 2.877150271	0.50 , 2.556528768
0.12 , 2.978235267	0.34 , 2.814647527	0.52 , 2.511207368
0.13 , 2.974422493	0.35 , 2.802707596	0.55 , 2.435113067
0.15 , 2.965844291	0.36 , 2.790300202	0.56 , 2.407196395
0.16 , 2.961073019	0.37 , 2.777410987	0.58 , 2.346719806
0.17 , 2.955975971	0.38 , 2.764024404	0.59 , 2.313782367
0.19 , 2.944789895	0.41 , 2.720704268	0.61 , 2.241184992
0.20 , 2.938692802	0.43 , 2.688985433	0.62 , 2.200750833

Table 2. Numerical real roots of the weak lens equations (4.46) and (4.49).

$\beta, \theta_p^*, \theta_s^*$	$\beta, \theta_p^*, \theta_s^*$
0.00 , -1.219460255 , -1.219460255	0.80 , -0.7348519977 , -1.499588276
0.05 , -1.199838645 , -1.238509312	0.85 , -0.6784510032 , -1.518118111
0.10 , -1.179561633 , -1.257061036	0.90 , -0.6155144835 , -1.537146134
0.15 , -1.158536660 , -1.275184787	0.95 , -0.5450735811 , -1.556736024
0.20 , -1.136659015 , -1.292945557	1.00 , -0.4661318364 , -1.576949887
0.25 , -1.113808946 , -1.310405245	1.05 , -0.3777428132 , -1.597847534
0.30 , -1.089848258 , -1.327623722	1.10 , -0.2791180071 , -1.619485734
0.35 , -1.064616274 , -1.344659714	1.15 , -0.1697572507 , -1.641917459
0.40 , -1.037925073 , -1.361571533	1.20 , -0.0495952394 , -1.665191148
0.45 , -1.009553834 , -1.378417648	1.25 , 0.0808230731 , -1.689350021
0.50 , -0.9792421704 , -1.395257127	1.30 , 0.2201081712 , -1.714431478
0.55 , -0.9466823165 , -1.412149940	1.35 , 0.3657571584 , -1.740466594
0.60 , -0.9115100856 , -1.429157137	1.40 , 0.5140618309 , -1.767479742
0.65 , -0.8732946679 , -1.446340903	1.45 , 0.6605639130 , -1.795488363
0.70 , -0.8315275959 , -1.463764491	1.50 , 0.8011244916 , -1.824502882
0.75 , -0.7856117298 , -1.481492040	1.55 , 0.9329783480 , -1.854526780